REPRESENTATION THEORY OF FINITE GROUPS

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1. Introduction

Broadly speaking, representation theory is the study of symmetries of vector spaces and the groups associated with those symmetries. The representation theory of groups allows us to study groups using the tools of linear algebra, and it encourages us to study groups from the perspective of symmetry, which was the perspective that led to the beginning of group theory in the first place. In this expository paper, we give a brief introduction to the representation theory of finite groups. While representation theory also exists for other structures, such as Lie groups, associative algebras, and quivers, the representations of finite groups are the most basic and also the most accessible for most. In Section 2, we lay the foundations for representation theory by proving some simple but powerful key results, such as Schur's Lemma and Maschke's Theorem. We also describe the representations of basic groups, like abelian groups and S_3. In Section 3, we introduce character theory. First introduced by Frobenius, group characters are a powerful tool for understanding not just representations, but groups themselves. In Section 4, we show Frobenius's divisibility theorem on the degrees of representations, and we introduce induced representations. Finally, in Section 5, we prove Burnside's $p^a q^b$-theorem on solvable groups using the tools of representation theory. Some knowledge of linear algebra and abstract algebra should be sufficient to understand this paper.

2. Representations of finite groups

In this section, G always denotes a finite group. We assume that all vector spaces are finite-dimensional and have ground field $\mathbb{C}$.

2.1. Basic theory.

Definition 2.1. A *representation* of a group G on a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ from G to the group of linear transformations of V. The dimension of V is also called the *degree* of the representation ρ.

Sometimes, we also call V itself a representation of G. Similarly, we often write gv directly instead of writing the less ambiguous $\rho(g)(v)$.

Naturally, when considering a new mathematical structure, we also wish to consider its substructures and maps between different structures.

Definition 2.2. A *subrepresentation* W of V is a subspace such that $gw \in W$ for all $g \in G$.

Definition 2.3. A *homomorphism of representations* (also called a *G-linear map*) $\phi : V \to W$ is a map of vector spaces that commutes with every $g \in G$. In other words, we have

$$g \cdot \phi(v) = \phi(g \cdot w)$$

for all $v \in V$, $w \in W$, and $g \in G$.

To illustrate what representations look like, we give a few examples.

Example. Let G be any group, and let $V = \mathbb{C}$. Define $\rho : G \to \mathrm{GL}(V)$ by $\rho(g)(v) = v$ for all $g \in G$ and $v \in V$. This is the *trivial* representation of G.

Example. Let G be a group that acts on the left on a finite set X. Let V be the vector space consisting of formal sums of the elements $\{e_x : x \in X\}$. Then V is the *permutation* representation of G, given by

$$g \cdot \sum_{x \in X} a_x e_x = \sum_{x \in X} a_x e_{gx}.$$

Example. The *regular* representation of G is the permutation representation given by the action of G on itself.

2.2. **Constructions of representations.** We can construct new representations from old ones, just like how we can combine vector spaces into new ones through operations like the direct sum and the tensor product. For two representations V and W of G, their direct sum $V \oplus W$ is also a representation of G, given by

$$g \cdot (v, w) = (gv, gw).$$

Similarly, their tensor product $V \otimes W$ is also a representation, given by

$$g \cdot (v \otimes w) = gv \otimes gw.$$

Furthermore, the vector spaces V^* is also a representation. The dual representation V^* is given by

$$(g \cdot f)(h) = f(g^{-1}h),$$

for all $f \in V^*$. Confirming this is left as an exercise to the reader.

From the identification $\mathrm{Hom}(V, W) \cong V^* \otimes W$, we can also make $\mathrm{Hom}(V, W)$ into a representation. Let $\phi \in \mathrm{Hom}(V, W)$. Then the representation structure is given by

$$(g \cdot \phi)(v) = g\phi(g^{-1}v),$$

for all $v \in V$.

2.3. **Semisimplicity, Maschke's theorem, and Schur's Lemma.** To simplify our search for representations, we wish to restrict our focus to the "simplest" representations, or those that can be used as "building blocks" for all other representations. This is analogous to considering only simple groups in group theory. However, there are two notions of "simplicity" that we might use.

Definition 2.4. A representation V is *irreducible* if its only subrepresentations are 0 and itself. A representation V is *indecomposable* if it cannot be written as a direct sum of two nonzero subrepresentations.

The concept of irreducibility lines up with our understanding of, say, simple groups, which have no normal subgroups. However, the concept of indecomposability lines up more with our mission: to consider the basic building blocks of all representations. Clearly, irreducibility implies indecomposability, but the converse is not necessarily clear. However, for finite groups, the converse is true by the following theorem:

Theorem 2.5 (Maschke)**.** *Let G be a group. Let V be a representation of G, and let W be a subrepresentation of V. Then there is a complementary subrepresentation W' of V, so that $V = W \oplus W'$.*

Proof. Let U be an arbitrary subspace complementary to W as vector spaces. Let $\pi_0 : V \to W$ be the projection of V onto W induced by the decomposition $V = W \oplus U$. Define the map

$$\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v)) : V \to V.$$

Since π_0 is a projection onto W, and W is invariant under action by G, the image of π is contained within W. Furthermore, if $w \in W$, we have that $g^{-1}w \in W$ as well. Thus, $\pi_0(g^{-1}w) \in W$ too, since π_0 is a projection onto W. Finally, since $g(\pi_0(g^{-1}w))$, the map π must also be a projection onto W.

We claim that π is also a G-linear map. Let $h \in G$. Then

$$\pi(hv) = \sum_{g \in G} g\pi_0 g^{-1}(hv) = \sum_{g \in G} hg\pi_0 g^{-1}(v) = h\pi(v).$$

Therefore, since π is G-linear, the kernel of π must be invariant under action by G. Let $W' = \ker \pi$. Then V decomposes into $W \oplus W'$ as a representation. $\square$

Corollary 2.6. *Any representation is semisimple, i.e. it can be written as a direct sum of irreducible representations.*

We can also prove Schur's Lemma, another powerful result about representations that characterizes maps between representations.

Lemma 2.7 (Schur)**.** *Let V and W be representations of a group G, and let ϕ be a homomorphism of representations from V to W.*

(i) If V is irreducible, then either $\phi = 0$ or ϕ is injective.
(ii) If W is irreducible, then either $\phi = 0$ or ϕ is surjective.
(iii) If $V = W$, and V is irreducible, then $\phi = \lambda I$ for some scalar $\lambda \in \mathbb{C}$, where I is the identity.

Parts (i) and (ii) imply that if both V and W are irreducible, ϕ is either 0 or an isomorphism.

Proof.

(i) The kernel $\ker \phi$ is a subrepresentation of ϕ (one can check that it satisfies the properties of a subrepresentation). Thus, if V is irreducible, it must either be 0, in which case ϕ is injective, or V, in which case $\phi = 0$.
(ii) The image $\operatorname{im} \phi$ is similarly a subrepresentation of ϕ. Therefore, if W is irreducible, it is either 0, in which case $\phi = 0$, or all of W, in which case ϕ is surjective.
(iii) Since $\mathbb{C}$ is algebraically closed, $\phi : V \to V$ must have an eigenvalue λ. By (i) and (ii), ϕ must either be 0 or an isomorphism. Since $\phi - \lambda I$ has a nonzero determinant, it cannot be an isomorphism, so it must be 0. Hence, $\phi = \lambda I$.

$\square$

Schur's Lemma implies that the decomposition given in Corollary 2.6 is unique up to order.

Corollary 2.8. *Let V be a representation of a group G. Then there is a decomposition*

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where $V_1, \ldots, V_k$ are distinct irreducible representations. This decomposition is unique up to ordering.

Proof. Let V have two decompositions into (not necessarily distinct) irreducible representations $\bigoplus V_m$ and $\bigoplus W_n$. Then let $\phi_{m,n} : V_m \to W_n$ be the map formed by the composition of the inclusion of V_m into V and the projection from V onto W_n. Since the identity map on V is injective, for every m, there must be some n such that $\phi_{m,n}$ is not the zero map. By Schur's Lemma, it must be an isomorphism. Therefore, for each V_m, there is some W_n such that $V_m \cong W_n$, and vice versa. Thus, the decomposition is unique. $\square$

2.4. **Key questions.** The key results just introduced establish irreducible representations as the basic building blocks of all representations. Thus, given a group G, a natural question to ask is:

Question 2.9. *What are all the irreducible representations of G?*

Once we've found all the irreducible representations, we then extend our analysis to all representations of G:

Question 2.10. *Given an arbitrary representation V, what is its direct sum decomposition into irreducibles?*

Answering these questions takes quite some work for each group individually, since we have no tools (yet) that can describe the irreducible representations of all groups that well. Examples for abelian groups and S_3 can be found in [**?**]. A summary will be given here.

Let G be an abelian group. For all $g \in G$, $\rho(g) : V \to V$ is a map of representations, since it commutes with all other $\rho(h)$ by commutativity in G. Therefore, by Schur's Lemma, each element g acts on V by scalar multiplication, so every subspace of V is invariant under G. Therefore, all irreducible representations of G are 1-dimensional.

Describing the representations of S_3 takes a little more work. Two 1-dimensional irreducible representations exist: the trivial representation U and the *alternating* representation U', in which gv is defined by $\mathrm{sgn}(g)v$. Since S_3 is naturally a permutation group, there is a permutation representation isomorphic to $\mathbb{C}^3$. However, this representation is not irreducible, since the subspace spanned by $(1, 1, 1)$ is invariant under G. The complementary subspace, however, is invariant, and it is called the *standard* representation (denoted V). Showing that these are the only irreducible representations takes more work. For an arbitrary representation of S_3, decomposing it into these irreducible representations involves looking at the eigenvalues of certain elements of S_3 in the representation. However, we will not go into detail here, and instead refer the reader to [**?**].

2.5. **The group algebra $\mathbb{C}[G]$.** For any group G, we can define the group algebra $\mathbb{C}[G]$ to be the algebra with basis vectors $\{e_g : g \in G\}$ and multiplication law $e_g e_h = e_{gh}$. One can see that representations of a finite group G are equivalent to left $\mathbb{C}[G]$-modules. For now, this identification is merely another way to think about group representations. In particular, it will later provide us with an alternative formulation for the induced representation that may be easier to work with.

3. Character theory

3.1. **Basic character theory.** Finding ad hoc methods to find the representations of an arbitrary group is rather difficult. A solution to this is character theory, which provides information about arbitrary representations for any group. It turns out that looking at eigenvalues is an effective way to describe the representations of a group. While considering all eigenvalues would be difficult to handle, we can make a key observation. Since knowing the eigenvalues λ_i of an element $g \in G$ gives us the eigenvalues λ_i^k of the element g^k, we can just consider the *sums* of eigenvalues $\sum \lambda_i$. If we know these sums for all $g \in G$, we should also have significant information about the eigenvalues themselves, without actually knowing them. This motivates the following definition.

Definition 3.1. Let V be a representation of a group G. Its *character* $\chi_V : G \to \mathbb{C}$ is a function defined by
$$\chi_V(g) = \mathrm{Tr}(g),$$
the trace of g in V.

By properties of the trace, the character χ_V is also a class function, i.e. $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $g, h \in G$.

Proposition 3.2. *Let V and W be representations of a group G. Then:*
$$\begin{aligned} \chi_{V\oplus W} &= \chi_V + \chi_W, \\ \chi_{V\otimes W} &= \chi_V \cdot \chi_W, \\ \textit{and } \chi_{V^*} &= \overline{\chi}_V. \end{aligned}$$

Proof. We prove these formulas by considering the eigenvalues of an element $g \in G$. Let g have eigenvalues $\{\lambda_i\}$ in V and $\{\mu_j\}$ in W. Then, in $V \oplus W$, g has eigenvalues $\{\lambda_i\} \cup \{\mu_j\}$, so $\chi_{V\oplus W} = \chi_V + \chi_W$. In $V \otimes W$, g has eigenvalues $\{\lambda_i \cdot \mu_j\}$, so $\chi_{V\otimes W} = \chi_V \chi_W$. In V^*, the eigenvalues of g are instead $\{\lambda_i^{-1}\}$. Since all the eigenvalues are $|G|$th roots of unity, the eigenvalues are equal to $\{\overline{\lambda}_i\}$. Therefore, $\chi_{V^*} = \overline{\chi}_V$. □

Since the character is a class function, we might want to express the information about the values of characters of representations on the conjugacy classes of a group. We can do so in a *character table*, where columns represent conjugacy classes and rows represent distinct irreducible representations. The entries of a character table are the values of the character of the corresponding representation on the corresponding conjugacy class. For example, the character table for S_3 (although we have not shown it) is

S_3	1	(12)	(123)
trivial U	1	1	1
alternating U'	1	-1	1
standard V	2	0	-1

Here, we can make an interesting observation: if we treat the values of the characters $\chi_U, \chi_{U'}, \chi_V$ on the conjugacy classes as vectors in $\mathbb{C}^3$, they are linearly independent. Thus, they form a basis, so any representation W of S_3 admits a unique decomposition into the direct sum of copies of U, U', and V, since the character of W can be expressed uniquely as a sum of the characters of the irreducible representations. In fact, a stronger version of this result holds for *all* groups, which we will see in the next section.

3.2. **Key results.** To find the multiplicity of the trivial representation in an arbitrary representation, we begin by considering the following morphism.

Proposition 3.3. *Let V be a representation of a group G, and let V^G be the direct sum of the trivial factors in the decomposition of V. Then the map*

$$\phi = \frac{1}{|G|}\sum_{g\in G} g$$

is a projection of V onto V^G.

Proof. Suppose $v = \phi(w)$ for some $v, w \in V$. Then, for any $h \in G$, we have

$$hv = \frac{1}{|G|}\sum_{g\in G} hgw = \frac{1}{|G|}\sum_{g\in G} gw = v.$$

Therefore, $\operatorname{im}\phi \subseteq V^G$.

Now, suppose that $v \in V^G$. Then

$$\phi(v) = \frac{1}{|G|}\sum_{g\in G} gv = \frac{1}{|G|}\sum_{g\in G} v = v,$$

so ϕ fixes V^G. Therefore, ϕ is a projection from V to V^G. □

Corollary 3.4. *Let V be a representation of a group G. Then the multiplicity of the trivial representation in V is*

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g).$$

Proof. The multiplicity of the trivial representation is just the trace of ϕ, which is

$$\operatorname{Tr}(\phi) = \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}(g) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g).$$

□

However, we wish to find the multiplicities of any irreducible representation, not just the trivial representation, in an arbitrary representation. Fortunately, the above claims can give such a result. Suppose that we wish to find the multiplicity of the irreducible representation V in another representation W. Let the decomposition of W into irreducibles be $\bigoplus W_i^{\oplus a_i}$. Letting $\operatorname{Hom}_G(A, B)$ denote the space of G-linear maps from A to B, we have

$$\operatorname{Hom}_G(V, W) \cong \operatorname{Hom}_G(V, \bigoplus W_i^{\oplus a_i}) \cong \bigoplus \operatorname{Hom}_G(V, W_i)^{\oplus a_i}.$$

By Schur's Lemma, $\operatorname{Hom}_G(V, W_i)$ is 1 when $V \cong W_i$ and 0 when $V \ncong W_i$, since V and W_i are irreducible for all i. By the above equality, we also have

$$\dim \operatorname{Hom}_G(V, W) = \dim \bigoplus \operatorname{Hom}_G(V, W_i)^{\oplus a_i} = \sum a_i(\dim \operatorname{Hom}_G(V, W_i)) = \sum_{V\cong W_i} a_i.$$

The last term is equal to the multiplicity of V in W, so we can find the multiplicity of V in W by computing $\dim \operatorname{Hom}_G(V, W)$.

Recall that for representations V and W, the space $\operatorname{Hom}(V, W)$ admits a representation structure by $(g \cdot \phi)(v) = g\phi(g^{-1}v)$ for all $\phi \in \operatorname{Hom}(V, W)$. One can check from this definition that the subspace of G-linear maps in $\operatorname{Hom}(V, W)$ is precisely the space of maps that is fixed by G, which is also the direct sum of the copies of the trivial representation in $\operatorname{Hom}(V, W)$. Thus, Corollary 3.4 gives us a way to calculate $\dim \operatorname{Hom}_G(V, W)$. In particular, the dimension is equal to

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}(V,W)}(g).$$

We can find $\chi_{\operatorname{Hom}(V,W)}$ by considering the identification $\operatorname{Hom}(V, W) \cong V^* \otimes W$. By this isomorphism, we have that

$$\chi_{\operatorname{Hom}(V,W)} = \overline{\chi_V} \chi_W.$$

Thus, the multiplicity of V in W can be found by calculating

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

We can also express this differently. Let $C(G)$ be the space of complex-valued class functions on G, and define a Hermitian inner product on $C(G)$ by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

Thus,

Theorem 3.5. *The characters of the irreducible representations of G are orthonormal with respect to this inner product.*

Proof. Fix V and W to be irreducible representations. Then (χ_V, χ_W) must be 1 if $V \cong W$ and 0 if $V \ncong W$ by the multiplicity formula. □

This important theorem gives us some powerful corollaries.

Corollary 3.6. *A representation is uniquely determined by its character.*

Proof. Let V be an arbitrary representation. If $V \cong \bigoplus V_i^{\oplus a_i}$, where the V_i are distinct irreducible representations, then $\chi_V = \sum a_i \chi_{V_i}$. Since the χ_{V_i} are orthonormal, they are also linearly independent, so this decomposition into irreducible characters corresponds uniquely to a decomposition into irreducible representations, which is unique by Corollary 2.8. □

Corollary 3.7. *Let V be a representation, and let its decomposition into distinct irreducibles be $\bigoplus V_i^{\oplus a_i}$. Then $(\chi_V, \chi_V) = \sum a_i^2$.*

Proof. We can write $\chi_V = \sum a_i \chi_{V_i}$. Then the inner product is equal to

$$\sum a_i a_j (\chi_{V_i}, \chi_{V_j}.$$

The terms where $i \neq j$ vanish, leaving only the terms of the form $a_i^2(\chi_{V_i}, \chi_{V_i})$. Since the V_i are irreducible, these terms are equal to a_i^2, giving us the desired equality. □

As a result of this corollary, a representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.

We also have the following theorem about the dimensions of irreducible representations.

Theorem 3.8. *Let G be a group. Then*

$$\sum_i (\dim V_i)^2 = |G|,$$

where the sum is over all irreducible representations of G.

Proof. Consider the regular representation R of G. We can find χ_R by considering the matrices of each element $g \in G$. If $g = e$, then g acts as the identity, so $\chi_R(g) = \dim R = |G|$. If $g \neq e$, then its matrix representation has all nonzero entries off the main diagonal. Thus, $\chi_R(g) = 0$ for all $g \neq e$. Now, we can find the multiplicities a_i of the irreducible representations V_i in R. We have

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|}\sum_{g \in G} \overline{\chi_{V_i}(g)}\chi_R(g) = \frac{1}{|G|}\chi_{V_i}(e) \cdot |G| = \dim V_i.$$

Therefore, each irreducible representation V_i appears in R a total of $\dim V_i$ times. As a result, $\dim R = \sum_i (\dim V_i)^2$. But $\dim R$ also equals $|G|$, so

$$|G| = \sum_i (\dim V_i)^2.$$

□

3.3. **Column orthogonality, characters form a basis.** In this section, we show two more main results about the characters of G: the orthogonality of the columns of a character table, and the result that the characters form an orthonormal basis for $C(G)$.

Theorem 3.9. *For $g \in G$,*

$$\sum_{\chi} \overline{\chi(g)}\chi(g) = \begin{cases} \frac{|G|}{c(g)} & \text{if } [g] = [h] \\ 0 & \text{if } [g] \neq [h], \end{cases}$$

where the sum is over all irreducible characters, $c(g)$ is the number of elements in the conjugacy class of g, and $[g]$ denotes the conjugacy class of g.

Proof. Recall that the orthonormality of characters implies that

$$\frac{1}{|G|}\sum_{g \in G} \overline{\chi_i(g)}\chi_j(g) = \delta_{ij}.$$

Taking this sum over all conjugacy classes C yields

$$\frac{1}{|G|}\sum_{C} |C|\overline{\chi_i(C)}\chi_j(C) = \delta_{ij}.$$

Now, let matrix T have entries (t_{ij}) with $t_{ij} = \overline{\chi_i(C_j)}$, and let matrix U have entries (u_{ij}) with $u_{ij} = |C_i|\chi_j(C_i)$. The above relations then imply that

$$\begin{aligned} \delta_{ij} &= \frac{1}{|G|}\sum_k (\overline{\chi_i(C_k)})(|C_k| \cdot \chi_j(C_k)) \\ &= \frac{1}{|G|} = \sum_k t_{ik}u_{kj} \\ &= \frac{1}{|G|}(TU)_{ij}. \end{aligned}$$

Therefore, $TU = |G| \cdot I$, so $UT = |G| \cdot I$ as well. Expanding the expression for $(UT)_{ij}$ yields

$$(UT)_{ij} = \sum_k u_{ik}t_{kj} = \sum_k |C_i|\chi_k(C_i)\overline{\chi_k(C_j)}.$$

Since $(UT)_{ij} = |G|\delta_{ij}$, we have two cases:

(1) If $i = j$, then we have the equality

$$|G| = \sum_k |C_i| \chi_k(C_i) \overline{\chi_k(C_i)}.$$

Therefore, we have

$$\frac{|G|}{|C_i|} = \sum_\chi \chi(C_i) \overline{\chi(C_i)},$$

as desired.

(2) Otherwise, if $i \neq j$, then the left-hand side is zero, so we have

$$0 = \frac{1}{|G|} \sum_\chi \chi_(C_i) \overline{\chi(C_j)},$$

as desired.

□

Now, we move on to prove our final result directly relating to characters.

Proposition 3.10. *The irreducible characters of G form an orthonormal basis for $C(G)$. Equivalently, the number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. We use the following lemma, a proof of which can be found in [**?**].

Lemma 3.11. *Let $\alpha : G \to \mathbb{C}$ be a complex function for a group G. For any representation V of G, define*

$$\phi_{\alpha,V} = \sum \alpha(g) \cdot g : V \to V.$$

Then $\phi_{\alpha,V}$ is a G-linear map if α is a class function.

To prove our original result, suppose that α is a class function such that $(\alpha, \chi_V) = 0$ for all irreducible characters χ_V. We wish to show that $\alpha = 0$. Consider the endomorphism $\phi_{\alpha,V}$. Since it is a G-linear map from V to V, it must be equal to λI for some scalar $\lambda \in \mathbb{C}$ by Schur's Lemma. Letting $n = \dim V$, we have

$$\begin{aligned} \lambda &= \frac{1}{n} \cdot \mathrm{Tr}(\phi_{\alpha,V}) \\ &= \frac{1}{n} \sum \alpha(g) \chi_V(g) \\ &= \frac{absG}{n} \overline{(\alpha, \overline{\chi_V})} \\ &= \frac{absG}{n} \overline{(\alpha, \chi_{V^*})} \\ &= 0. \end{aligned}$$

Therefore, $\sum \alpha(g) \cdot g = 0$ for all representations V of G. Let $V = R$, the regular representation of G. Then the elements g of R, when thought of as elements of $\mathrm{End}(R)$, are linearly independent. Therefore, $\alpha(g) = 0$, as desired. □

4. Further results on representations

In this section, we show some further results of representations. The first is Frobenius' divisibility theorem, which states that the dimension of an irreducible representation must divide the order of its group. The second is a description of restricted and induced representations, which are ways to switch between representations of a group G and one of its subgroups H. Our discussion of induced representations culminates in the proof of Frobenius reciprocity, which establishes a relationship between induction and restriction.

4.1. Frobenius' divisibility theorem.

Theorem 4.1 (Frobenius). *Let V be an irreducible representation of a group G. Then $\dim V$ divides $|G|$.*

To prove this, we first state some results about algebraic integers.

Definition 4.2. A number $z \in \mathbb{C}$ an *algebraic integer* if z is the root of a monic polynomial with integer coefficients. The set of algebraic integers is denoted $\mathbb{A}$.

We will not prove the following claims, but proofs of them can be found in [?].

Proposition 4.3. *(1) $\mathbb{A}$ is a ring.*
(2) $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.
(3) Let the algebraic conjugates of an algebraic integer α be the roots of the minimal polynomial of α. If $\alpha_1, \dots, \alpha_m$ are algebraic integers, then all algebraic conjugates of $\alpha_1 + \cdots + \alpha_m$ are of the form $\alpha_1' + \cdots + \alpha_m'$, where each α_i' is an algebraic conjugate of α_i.
(4) Let G be a group. Any element of the algebra $\mathbb{Z}[G]$ satisfies a monic equation with integer coefficients.

We first prove the following lemma.

Lemma 4.4. *Let $C_1, C_2, \dots, C_n$ be the conjugacy classes of G, and g_{C_i} be representatives of C_i. Define*

$$\lambda_i = \chi_V(g_{C_i}) \frac{|C_i|}{\dim V}.$$

The numbers λ_i are algebraic integers for all i.

Proof. Let C be a conjugacy class in G, and set $P = \sum_{h \in C} h$. Since P is a G-linear map, by Schur's Lemma, it acts on V by scalar multiplication for some scalar λ. By part (iv) of 4.3, this λ is an algebraic integer.

We can also calculate $\mathrm{Tr}(P)$ in V. By the definition of P, this equals $|C|\chi_V(g_C)$. Since P is equal to scalar multiplication by λ, $\mathrm{Tr}(P)$ also equals $\lambda \dim V$. Thus, $\lambda = \chi_V(g_C)\frac{|C|}{\dim V}$ is an algebraic integer. □

Now, we can prove the original theorem.

Proof. Consider the number

$$\mu = \sum_i \lambda_i \overline{\chi_V(g_{C_i})}.$$

Since the eigenvalues of g_{C_i} are roots of unity, $\chi_V(g_{C_i})$ is a sum of roots of unity and therefore an algebraic integer. Since λ_i is also an algebraic integer, and $\mathbb{A}$ is a ring, μ is in $\mathbb{A}$.

We can expand μ using the definition of λ_i. We have

$$\mu = \sum_i \lambda_i \overline{\chi_V(g_{C_i})} = \sum_i \frac{|C_i| \chi_V(g_{C_i}) \overline{\chi_V(G_{C_i})}}{\dim V}.$$

Since χ_V is a class function, this is equal to

$$\sum_{g \in G} \frac{\chi_V(g)\overline{\chi_V(g)}}{\dim V} = \frac{|G|}{\dim V}(\chi_V, \chi_V).$$

Since V is irreducible, $(\chi_V, \chi_V) = 1$, so $\mu = \frac{|G|}{\dim V}$. Since this is an algebraic integer as well as rational, it must be an integer. Thus, $\dim V \mid |G|$. □

4.2. **Induced representations.** Suppose that H is a subgroup of G. Then any representation V of G naturally restricts to a representation of H. This restriction is denoted $\operatorname{Res}_H^G V$. The degree and character of $\operatorname{Res}_H^G V$ are equal to that of V itself.

We can also construct representations the other way. Let V be a representation of H. We then wish to construct a representation of G that retains some structure of the original representation. We provide two constructions of this representation, which is called the *induced representation* and denoted $\operatorname{Ind}_H^G V$.

The first construction is the same as that given in [**?**]. For each coset $\sigma \in G/H$, take a copy of V, written V^σ. In V^σ, let $g_\sigma v$ be the element corresponding to $v \in V$, where $\{g_\sigma\}$ is a system of coset representatives of H. Let $\operatorname{Ind}_H^G V = W = \bigoplus_{\sigma \in G/H} V^\sigma$, so that each element of W can be uniquely written as $w = \sum g_\sigma v_\sigma$, where $v_\sigma \in V$. Define the action of $g \in G$ on W by

$$g \cdot (g_\sigma v_\sigma) = g_\tau (h w_\sigma),$$

where $g g_\sigma = g_\tau h$. To verify that this defines a representation, we must show that

$$(g' \cdot g)(g_\sigma v_\sigma) = g' \cdot (g \cdot (g_\sigma v_\sigma))$$

for all $g, g' \in G$. Let $g' g_\tau = g_\rho h'$. Then

$$g' \cdot (g \cdot (g_\sigma v_\sigma)) = g' \cdot (g_\tau (h v_\sigma)) = g_\rho (h'(h v_\sigma)).$$

We also have $(g' \cdot g) \cdot g_\sigma = g' \cdot (g \cdot g_\sigma) = g' \cdot g_\tau \cdot h = g_\rho \cdot h' \cdot h$, so

$$(g' \cdot g) \cdot (g_\sigma v_\sigma) = g_\rho ((h' \cdot h) v_\sigma) = g_\rho (h'(h v_\sigma)).$$

Thus, this does indeed define an action of G on W.

We can also express $\operatorname{Ind}_H^G V$ in terms of the group algebra $\mathbb{C}[G]$. In fact, it is simply

$$\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V,$$

where G acts on the first factory by $g \cdot (e_{g'} \otimes v) = e_{gg'} \otimes v$.

To find the character of $\operatorname{Ind}_H^G V$, notice that $g \in G$ sends V^σ to $V^{g\sigma}$. Therefore, the trace only includes the cosets σ such that $g\sigma = \sigma$. This is equivalent to the condition that $sgs^{-1} \in H$ for $s \in \sigma$, so the character of $\operatorname{Ind} V$ is

$$\chi_{\operatorname{Ind} V} g = \sum_{g\sigma = \sigma} \chi_V (sgs^{-1}).$$

4.3. **Results on induced representations.**

Proposition 4.5. *Let $H \leq K \leq G$ be subgroups. If V is a representation of H, then*

$$\operatorname{Ind}_H^G V = \operatorname{Ind}_K^G (\operatorname{Ind}_H^K V).$$

Proof. This is most easily proved by using the tensor product construction of the induced representation. Using this definition, we have

$$\begin{aligned} \operatorname{Ind}_K^G (\operatorname{Ind}_H^K V) &= \mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V) \\ &= (\mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C}[K]) \otimes_{\mathbb{C}[H]} V \\ &= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \\ &= \operatorname{Ind}_H^G V. \end{aligned}$$

□

The following result, Frobenius reciprocity, establishes a relationship between Ind and Res.

Theorem 4.6 (Frobenius reciprocity)**.** *Let H be a subgroup of a group G. Let W be a representation of H and V a representation of G. Then*

$$\operatorname{Hom}_G(W, \operatorname{Res}_H^G V) \cong \operatorname{Hom}_H(\operatorname{Ind}_H^G W, V).$$

We can also express this result in category theoretic terms. Let $\mathbf{Rep}_G$ be the category of representations of G, whose morphisms are G-linear maps. Define $\mathbf{Rep}_H$ similarly for H. Then Ind is a functor from $\mathbf{Rep}_H$ to $\mathbf{Rep}_G$, and Res is a functor from $\mathbf{Rep}_G$ to $\mathbf{Rep}_H$. Frobenius reciprocity states that they are both left and right adjoint to each other.

Proof. Let $\phi : W \to \operatorname{Res} V$ be an H-linear homomorphism. We wish to show that it extends uniquely to a G-linear homomorphism ϕ' from $\operatorname{Ind} W$ to V. Write $\operatorname{Ind} W = \bigoplus_{\sigma \in G/H} W^\sigma$, and define the action of ϕ' on W^σ to be $g_\sigma \phi g_\sigma^{-1}$. We can also construct the inverse morphism. If $\psi : \operatorname{Ind} W \to V$ is a G-linear homomorphism, it clearly restricts to an H-linear homomorphism from W to $\operatorname{Res} V$. We leave it to the reader to show that these indeed constitute an isomorphism $\operatorname{Hom}_G(W, \operatorname{Res}_H^G V) \cong \operatorname{Hom}_H(\operatorname{Ind}_H^G W, V)$. ☐

Corollary 4.7. *Let W be a representation of H and V a representation of G. Then*

$$(\chi_{\operatorname{Ind} W}, \chi_V)_G = (\chi_W, \chi_{\operatorname{Res} V})_V.$$

Proof. We prove this for irreducible V, W. The left-hand side is equal to the multiplicity of V in $\operatorname{Ind} W$, which is equal to $\dim \operatorname{Hom}_G(\operatorname{Ind} W, V)$. The right-hand side is equal to the multiplicity of W in $\operatorname{Res} V$, or $\dim \operatorname{Hom}_G(W, \operatorname{Res} V)$. By Frobenius reciprocity, these dimensions are equal, so the inner products are also equal. ☐

Corollary 4.8. *Let W be a representation of H and V a representation of G. Then*

$$\operatorname{Ind} W \otimes V \cong \operatorname{Ind}(W \otimes \operatorname{Res} V).$$

Proof. Let $E = \operatorname{Ind} W \otimes V$ and $F = \operatorname{Ind}(W \otimes \operatorname{Res} V)$. To show that $E \cong F$, we show that for all representations U of G, we have $\operatorname{Hom}_G(E, U) \cong \operatorname{Hom}_G(F, U)$. This implies that E and F have the same decomposition into irreducibles, so they must be isomorphic.

By Frobenius reciprocity, we have

$$\operatorname{Hom}(\operatorname{Ind}(W \otimes \operatorname{Res} V), U) \cong \operatorname{Hom}(W \otimes \operatorname{Res} V, \operatorname{Res} U).$$

Repeatedly applying the tensor-hom adjunction and the isomorphism $\operatorname{Hom}(A, B) \cong A^* \otimes B$ yields the isomorphisms

$$\begin{aligned}
\operatorname{Hom}(W \otimes \operatorname{Res} V, \operatorname{Res} U) &\cong \operatorname{Hom}(W, \operatorname{Hom}(\operatorname{Res} V, \operatorname{Res} U)) \\
&\cong \operatorname{Hom}(W, (\operatorname{Res} V)^* \otimes \operatorname{Res} U) \\
&\cong \operatorname{Hom}(W, \operatorname{Res}(V^* \otimes U)).
\end{aligned}$$

By Frobenius reciprocity, we have

$$\operatorname{Hom}(W, \operatorname{Res}(V^* \otimes U)) \cong \operatorname{Hom}(\operatorname{Ind} W, V^* \otimes U).$$

Finally, by the lemmas applied previously, we have

$$\operatorname{Hom}(\operatorname{Ind} W, V^* \otimes U) \cong \operatorname{Hom}(\operatorname{Ind} W, \operatorname{Hom}(V, U)) \cong \operatorname{Hom}(\operatorname{Ind} W \otimes V, U).$$

Therefore, $\operatorname{Ind} W \otimes V \cong \operatorname{Ind}(W \otimes \operatorname{Res} V)$. ☐

5. Applications to group theory

In this section, we prove the following theorem of Burnside:

Theorem 5.1 (Burnside). *Any group of order $p^a q^b$, where p and q are primes and $a, b \geq 0$, is solvable.*

While a group theoretic proof of Burnside's theorem has been published [?] [?], they are much more complicated than Burnside's original proof, which involves representation theory. The proof written here is adapted from [?].

Lemma 5.2. *Let $\epsilon_1, \ldots, \epsilon_n$ be roots of unity such that $\frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n)$ is an algebraic integer. Then either $\epsilon_1 = \cdots = \epsilon_n$, or $\epsilon_1 + \cdots + \epsilon_n = 0$.*

Proof. Let $a = \frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n)$. Assume that not all ϵ_i are equal. Then $|a| < 1$. Since the algebraic conjugates of roots of unity are also roots of unity, we have $|a'| \leq 1$ for any algebraic conjugate a' of a. The product of all algebraic conjugates of a must be an integer, since a is an algebraic integer. Since this integer has absolute value < 1, it is 0, so $a = 0$. □

Using this lemma, we can prove the following theorem.

Theorem 5.3. *Let V be an irreducible representation of a group G, and let C be a conjugacy class of G such that $\gcd(|C|, \dim V) = 1$. Then for any $g \in C$, either $\chi_V(g) = 0$ or g acts on V by scalar multiplication.*

Proof. Let $n = \dim V$, and let $\epsilon_1, \ldots, \epsilon_n$ be the eigenvalues of g. Since they are roots of unity, $\chi_V(g)$ must be an algebraic integer. Because $\mathbb{A}$ forms a ring, $\frac{1}{n}|C|\chi_V(g)$ is also an algebraic integer. Also, since $\gcd(|C|, n) = 1$, we must have integers a, b such that $a|C| + bn = 1$. Thus,

$$\frac{a|C|\chi_V(g)}{n} + b\chi_V(g) = \frac{\chi_V(g)}{n} = \frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n)$$

is an algebraic integer. By Lemma 5.2, either all the eigenvalues of g are equal, or their sum is 0. In the first case, g acts by scalar multiplication. In the second case, $\chi_V(g) = 0$. □

We also prove the following lemma.

Lemma 5.4. *Let C be a conjugacy class of G with order p^k, where p is prime and $k > 0$. Then there is a nontrivial irreducible representation V, where $p \nmid \dim V$, such that $\chi_V(g) \neq 0$ for any $g \in C$.*

Proof. Since C is not the conjugacy class of the identity (as $k > 0$), we have

$$\sum_W \overline{\chi_W(e)}\chi_W(g) = 0$$

, where the sum is over all irreducible representations W, by column orthogonality. This implies that

$$\sum_W (\dim W)\chi_W(g) = 0.$$

Let U denote the trivial representation, let D denote the set of irreducible representations V such that $p \mid \dim V$, and let N denote the set of nontrivial irreducible representations V such that $p \nmid \dim V$. Then $U \cup D \cup N$ is the set of all irreducible representations of G, so we can split up the above sum accordingly. Thus,

$$\begin{aligned} 0 &= (\dim U)\chi_U(g) + \sum_{V \in D} (\dim V)\chi_V(g) + \sum_{V \in N} (\dim V)\chi_V(g) \\ &= 1 + \sum_{V \in D} (\dim V)\chi_V(g) + \sum_{V \in N} (\dim V)\chi_V(g). \end{aligned}$$

Assume that there is no $V \in N$ such that $\chi_V(g) = 0$. Then the sum $\sum_{V \in N} (\dim V)\chi_V(g)$ is 0, so by the above equation, $a = \sum_{V \in D} (\dim V)\chi_V(g) = -1$. But $\frac{a}{p}$ is an algebraic integer, since $p \mid \dim V$ for all $V \in D$. However, $\frac{-1}{p}$ cannot only an algebraic integer, since $p > 1$. Therefore, we have a contradiction, so there must be some $V \in N$ such that $\chi_V(g) = 0$. □

Now, we can prove the following result, which gives us most of Burnside's theorem.

Theorem 5.5. *Let G be a finite group, and let C be a conjugacy class in G of order p^k, where p is prime and $k > 0$. Then G is not simple.*

Proof. We can split up the irreducible representations of G again into U, D, and N, as we did in the proof of Lemma 5.4. By Lemma 5.4, there is some $V \in N$ such that $\chi_V(g) \neq 0$ for $g \in C$. But since $\gcd(|C|, \dim V) = 1$, by Theorem 5.3, g must act by scalar multiplication on V. Thus, every element in C must also act by scalar multiplication.

Let H be the subgroup of G generated by elements ab^{-1} for $a, b \in C$. It is normal, since for any $k \in G$,

$$kab^{-1}k^{-1} = kak^{-1}kb^{-1}k^{-1} = (kak^{-1})(kbk^{-1})^{-1}.$$

It must also act trivially on V. Since V is nontrivial, $H \neq G$. Furthermore, since $|C| > 1$, H cannot be trivial. Thus, H is a proper nontrivial normal subgroup of G, so G is not simple. □

We can now prove Burnside's Theorem.

Proof. Assume that Burnside's Theorem is false. Let G be the smallest nonsolvable group of order $p^a q^b$. Assume that G has a proper nontrivial normal subgroup N. Then N and G/N must be solvable, since G is the smallest nonsolvable group of its form. But then G must be solvable, so G cannot have such a subgroup. Thus, G is simple.

By Theorem 5.5, G cannot have a conjugacy class of order p^k or q^k for $k \geq 1$. Thus, the order of any conjugacy class of G must either be 1 or divisible by pq. But there is only one conjugacy class of order 1, so the sum of the orders of the conjugacy classes must be $1 \pmod{p}$ and $1 \pmod{q}$. However, the sum of the orders of the conjugacy classes must be equal to $|G|$, which is $p^a q^b$. This is a contradiction, so Burnside's Theorem must be true. □